

On p -adic colligations and 'rational maps' of Bruhat-Tits trees

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Consider matrices of order $k + N$ over p -adic field determined up to conjugations by elements of GL over p -adic integers. We define a product of such conjugacy classes and construct the analog of characteristic functions (transfer functions), they are maps from Bruhat-Tits trees to Bruhat-Tits buildings. We also examine categorical quotient for usual operator colligations.

1 Introduction

1.1. Notation. Denote by $1 = 1_\alpha$ the unit matrix of order α . Below K is an infinite field², \mathbb{K} is a locally compact non-Archimedean field, $\mathbb{O} \subset \mathbb{K}$ is the ring of integers. In both cases we keep in mind the p -adic fields. Let $\text{Mat}(n) = \text{Mat}(n, K)$ be the space of matrices of order n over K , $\text{GL}(n, K)$ the group of invertible matrices of order n . We say that an $\infty \times \infty$ matrix g is *finite* if $g - 1$ has finite number of nonzero matrix elements³. Denote by $\text{Mat}(\infty) = \text{Mat}(\infty, \mathbb{K})$ the space of finite $\infty \times \infty$ matrices, by $\text{GL}(\infty, K)$ the group of finite invertible finite matrices.

1.2. Colligations. Consider the space $\text{Mat}(\alpha + \infty, K)$ of finite block complex matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $(\alpha + \infty) \times (\alpha + \infty)$. Represent the group $\text{GL}(\infty, K)$ as the group of matrices of the form $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}$ of size $\alpha + \infty$. Consider conjugacy classes of $\text{Mat}(\alpha + \infty, K)$ with respect to $\text{GL}(\infty, K)$, i.e., matrices determined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}^{-1}, \quad \text{where } u \in \text{GL}(\infty, \mathbb{K}). \quad (1.1)$$

We call conjugacy classes by *colligations* (another term is '*nodes*'). Denote by

$$\text{Coll}(\alpha) = \text{Coll}(\alpha, K)$$

the set of equivalence classes. There is a natural multiplication on $\text{Coll}(\alpha, K)$, it is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 & q \\ 0 & 1 & 0 \\ r & 0 & t \end{pmatrix} = \begin{pmatrix} ap & b & aq \\ cp & d & cq \\ r & 0 & t \end{pmatrix}. \quad (1.2)$$

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²We prefer infinite fields, otherwise the rational function (1.6) is not well defined.

³Thus 1_∞ is finite and 0 is not finite

The size of the last matrix is

$$\alpha + \infty + \infty = \alpha + \infty.$$

The following statement is straightforward.

Proposition 1.1 a) *The \circ -multiplication is a well-defined operation*

$$\text{Coll}(\alpha) \times \text{Coll}(\alpha) \rightarrow \text{Coll}(\alpha).$$

b) *The \circ -multiplication is associative.*

There is a way to visualize this multiplication. We write the following 'perverse' equation for eigenvalues:

$$\begin{pmatrix} q \\ x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ \lambda x \end{pmatrix}, \quad (1.3)$$

where $\lambda \in K$. Equivalently,

$$q = ap + \lambda bx; \quad (1.4)$$

$$x = cp + \lambda dx. \quad (1.5)$$

We express x from (1.5),

$$x = (1 - \lambda d)^{-1} cp,$$

substitute it to (1.4), and get

$$q = \chi_g(\lambda)p,$$

where $\chi_g(\lambda)$

$$\chi_g(\lambda) = a + \lambda b(1 - \lambda d)^{-1}c \quad (1.6)$$

is a rational function $K \rightarrow \text{Mat}(\alpha)$. It is called *characteristic function of g* . The following statement is obvious.

Proposition 1.2 *If g_1 and g_2 are contained in the same conjugacy class, then their characteristic functions coincide.*

The next statement can be verified by a straightforward calculation (for a more reasonable proof, see below Theorem 4.2).

Theorem 1.3

$$\chi_{g \circ h}(\lambda) = \chi_g(\lambda)\chi_h(\lambda).$$

Theorem 1.4 *Let K be algebraically closed. Then any rational map $K \rightarrow \text{Mat}(\alpha, K)$ regular at 0 has the form $\chi_g(\lambda)$ for a certain $g \in \text{Mat}(\infty, K)$.*

See. e.g., [5], Theorem 19.1.

1.3. Origins of the colligations. The colligations and the characteristic functions appeared independently in spectral theory of non-self-adjoint operators (M.S.Livshits, 1946) and in system theory, see e.g., [13], [14], [28], [15], [9], [30], [5], [3], [8]. It seems that in both cases there are no visible reason to pass to p -adic case.

However, colligations and colligation-like objects arose by independent reasons in representation theory of infinite-dimensional classical groups, see [25], [16].

First, consider a locally compact non-Archimedean field \mathbb{K} and the double cosets

$$M = \mathrm{SL}(2, \mathbb{O}) \backslash \mathrm{SL}(2, \mathbb{K}) / \mathrm{SL}(2, \mathbb{O}).$$

The space of functions on M is a commutative algebra with respect to the convolution on $\mathrm{SL}(2, \mathbb{K})$. This algebra acts in the space of $\mathrm{SL}(2, \mathbb{O})$ -fixed vectors of any unitary representation of $\mathrm{SL}(2, \mathbb{K})$. Next (see Ismagilov [10], 1967), let us replace \mathbb{K} by a non-Archimedean non-locally compact field (i.e., the residue field is infinite [10] or the norm group is non-discrete [12]). Then there is no convolution, however double cosets have a natural structure of a semigroup, and this semigroup acts in the space of $\mathrm{SL}(2, \mathbb{O})$ -fixed vectors of any unitary representation of $\mathrm{SL}(2, \mathbb{K})$. In particular, this allows to classify all irreducible unitary representations of $\mathrm{SL}(2, \mathbb{K})$ having a non-zero $\mathrm{SL}(2, \mathbb{O})$ -fixed vector.

It appeared that these phenomena (semigroup structure on double cosets $L \backslash G / L$ for infinite dimension groups^{4,5} and actions of this semigroup in the space of L -fixed vectors) are quite general, see, e.g., [24], [25], [26], [16], [19], [18].

In [23] there was proposed a way to construct representations of infinite-dimensional p -adic groups, in particular there appeared semigroups of double cosets and p -adic colligation-like structures. The present work is a simplified parallel of [23]. If we look to the equivalence (1.1), then a p -adic field is an representative of non-algebraically closed fields. However, [23] suggests another equivalence,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}^{-1}, \quad \text{where } u \in \mathrm{GL}(\infty, \mathbb{O}) \quad (1.7)$$

(we conjugate by the group $\mathrm{GL}(\infty, \mathbb{O})$ of integer matrices). Below we construct analogs of characteristic functions for this equivalence and get 'rational' maps from Bruhat-Tits trees to Bruhat-Tits buildings (for $\alpha = 1$ we get maps from trees to trees), the characteristic function (1.6) is its boundary values on the absolute of the tree.

⁴There is a elementary explanation initially proposed by Olshanski: such semigroups are limits of Hecke-type algebras at infinity. For more details, see [22]

⁵Conjugacy classes are special cases of double cosets, conjugacy classes G with respect to L are double cosets $L \backslash (G \times L) / L$, where L is embedded to $G \times L$ diagonally, $l \mapsto (l, l)$.

It is interesting that maps of this type arise in theory of Berkovich rigid analytic spaces⁶, see [1], [2], [4]. However I do not understand links between two points of view. For instance, we show that any rational map of a projective line $\mathbb{P}\mathbb{Q}_p^1$ to itself admits a continuation to the Bruhat-Tits tree, and such continuations are enumerated by the set $\mathrm{GL}(\infty, \mathbb{Q}_p)/\mathrm{GL}(\infty, \mathbb{O})$. In Berkovich theory continuations of this type are canonical.

1.4. Structure of the paper. In Section 2 we consider characteristic functions over algebraically closed field. We discuss categorical quotient $[\mathrm{Coll}(\alpha)]$ of $\mathrm{Coll}(\alpha)$ with respect to the equivalence (1.1), the main statement here is Theorem 2.10. In Section 3 we examine the case $\alpha = 1$. We show that the semigroup $[\mathrm{Coll}(1)]$ is commutative. Also we show that for non-algebraically closed field any rational function $K \rightarrow K$ is a characteristic function.

In Section 4 we consider p -adic fields and introduce characteristic functions for conjugacy classes of $\mathrm{GL}(\alpha + \infty, \mathbb{Q}_p)$ by $\mathrm{GL}(\infty, \mathbb{O}_p)$.

In Section 5 we briefly discuss conjugacy classes of $\mathrm{GL}(\alpha + m\infty, \mathbb{Q}_p)$ with respect to $\mathrm{GL}(\infty, \mathbb{O}_p)$.

2 Formalities. Algebraically closed fields

In this section K is an algebraically closed field. For exposition of basic classical theory, see the textbook of Dym [5], Chapter 19. See more in [15], [9], [30], [31]. Our 'new' element is the categorical quotient⁷ (in a wider generality it was discussed in [21]).

Denote by $\mathbb{P}K^1$ the projective line over K . For an even dimensional linear space W denote by $\mathrm{Gr}(W)$ the Grassmannian of subspaces of dimension $\frac{1}{2} \dim V$.

2.1. Colligations. Fix $\alpha \geq 0$, $N > 0$. Consider the space of matrices $\mathrm{Mat}(\alpha + N, K)$, we write its elements as block matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider the group $\mathrm{GL}(N, K)$, we represent its elements as block matrices $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}$. Denote by $\mathrm{Coll}_N(\alpha, K)$ the space of conjugacy classes of $\mathrm{Mat}(\alpha + N)$ with respect to $\mathrm{GL}(N, K)$, see (1.1). Denote by $[\mathrm{Coll}_N(\alpha, K)]$ the categorical quotient (see, e.g., [27]), i.e., the spectrum of the algebra of $\mathrm{GL}(N, K)$ -invariant polynomials on $\mathrm{Mat}(\alpha + N)$.

2.2. Characteristic function. For an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{Mat}(\alpha + N)$ we assign the characteristic function

$$\chi_g(\lambda) = a + \lambda b(1 - \lambda d)^{-1}c, \quad \lambda \text{ ranges in } K. \quad (2.1)$$

⁶In Berkovich theory objects are larger than trees and buildings. However, our 'characteristic functions' admit extensions to these larger objects

⁷I have not met discussion of this topic, however sets of 'nonsingular points' of $\mathrm{Coll}(\alpha)$ and its completions were discussed in literature, see [9], [30].

If d is invertible, we extend this function to the point $\lambda = \infty$ by setting

$$\chi_g(\infty) = a - bd^{-1}c.$$

Passing to the coordinate $s = \lambda^{-1}$ on $\mathbb{P}K^1$, we get

$$\chi_g(s) = a + b(s - d)^{-1}c.$$

Theorem 2.1 *Any rational function $K \rightarrow \text{Mat}(\alpha, K)$ regular at 0 is a characteristic function of an operator colligation.*

See, e.g., [5], Theorem 19.1.

2.3. The characteristic function as a map $\mathbb{P}K^1 \rightarrow \text{Gr}(K^{2\alpha})$. See [15], [9], [31]. If λ_0 is a regular point of $\chi_g(\lambda)$, we consider its graph $\mathcal{X}_g(\lambda_0)$,

$$\mathcal{X}_g(\lambda_0) \subset K^\alpha \oplus K^\alpha.$$

Singularities of rational maps of $\mathbb{P}K^1$ to a projective variety $\text{Gr}(K^\alpha \oplus K^\alpha)$ are removable. Let us remove a singularity explicitly at a pole $\lambda = \lambda_0$. We can represent $\chi_g(\lambda)$ as

$$A(\lambda - \lambda_0) \begin{pmatrix} \frac{h_1}{(\lambda - \lambda_0)^{m_1}} & 0 & \cdots \\ 0 & \frac{h_2}{(\lambda - \lambda_0)^{m_2}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} B(\lambda - \lambda_0) + S(\lambda - \lambda_0), \quad (2.2)$$

where $A(\dots)$, $B(\dots)$ are polynomial functions $K \rightarrow \text{Mat}(\alpha)$, $A(0)$, $B(0)$ are invertible, the exponents m_i satisfy $m_1 \geq m_2 \geq \dots$, and $S(\lambda)$ is a rational functions $K \rightarrow \text{Mat}(\alpha)$ having zero of any prescribed order $M > 0$ (proof of this is a straightforward repetition of the Gauss elimination procedure). Denote by e_j the standard basis in K^α . Consider the subspace L in $K^\alpha \oplus K^\alpha$ generated by vectors

$$\begin{aligned} e_i \oplus 0, & \quad \text{for } m_i > 0; \\ h_j e_j \oplus e_j, & \quad \text{for } m_j = 0; \\ 0 \oplus e_l, & \quad \text{for } m_l < 0. \end{aligned}$$

Applying the operator $A(0) \oplus B(0)^{-1}$ to L we get $\chi_g(\lambda_0)$.

2.4. An exceptional divisor. A characteristic function is not sufficient for a reconstruction of a colligation. Indeed, consider a block matrix of size $\alpha + k + l$

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

Then characteristic the function is independent on e .

For $g \in \text{Coll}_N(\alpha)$ we define an additional invariant, a divisor⁸ $\Xi_g \subset \mathbb{P}K^1$ in the following way: Ξ_g as the divisor of zeros of the polynomial

$$p_g(\lambda) = \det(1 - \lambda d)$$

plus $\lambda = \infty$ with multiplicity $N - \deg p_g$. In the coordinate $s = \lambda^{-1}$ this divisor is simply the set of eigenvalues of d .

Proposition 2.2

$$\det \chi_g(\lambda) = \frac{\det \begin{pmatrix} a & -\lambda b \\ c & 1 - \lambda d \end{pmatrix}}{\det(1 - \lambda d)}.$$

PROOF. We apply the formula for the determinant of a block matrix. \square

Corollary 2.3 *The divisor Ξ_g contains the divisor of poles of $\det \chi_g(\lambda)$.*

Theorem 2.4 *For any rational function $K \rightarrow \text{Mat}(\alpha)$ regular at 0 there is a colligation g such that the divisor Ξ_g coincides with the divisors of poles of $\det \chi_g(\lambda)$.*

See [5], Theorem 19.8. Such colligations g are called *minimal*.

2.5. Invariants.

Theorem 2.5 *A point \mathbf{g} of the categorical quotient $[\text{Coll}_N(\alpha)]$ is uniquely determined by the characteristic function $\chi_{\mathbf{g}}(\lambda)$ and the divisor $\Xi_{\mathbf{g}}$.*

PROOF. Let us describe $\text{GL}(N, K)$ -invariants on $\text{Mat}(\alpha + N)$. A point $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(\alpha + N)$ can be regarded as the following collection of data:

- a) the matrix d ;
- b) α vectors (columns $c[j]$ of c);
- c) α covectors (rows $b[i]$ of b);
- d) scalars a_{ij} .

The algebra of invariants (see [29], Section 11.8.1) is generated by the following polynomials

$$b[i]d^k c[j], \tag{2.3}$$

$$\text{tr } d^k, \tag{2.4}$$

$$a_{ij}. \tag{2.5}$$

Expanding the characteristic function in λ ,

$$\chi_g(\lambda) = a + \sum_{k=0}^{\infty} \lambda^{k+1} b d^k c$$

⁸i.e. a finite set with multiplicities.

we get in coefficients all the invariants (2.3), (2.5). Expanding

$$\ln p_g(\lambda) = \ln \det(1 - \lambda d) = - \sum_{j=k}^{\infty} \frac{1}{k} \lambda^k \operatorname{tr} d^k,$$

we get all invariants (2.4). \square

Corollary 2.6 *Any point of $[\operatorname{Coll}_N(\alpha)]$ has a representative of the form*

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix},$$

where e is diagonal matrix and the colligation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is minimal.

2.6. \circ -product. Now we define the operation

$$\operatorname{Coll}_{N_1}(\alpha) \times \operatorname{Coll}_{N_2}(\alpha) \rightarrow \operatorname{Coll}_{N_1+N_2}(\alpha)$$

by the formula (1.2).

Theorem 2.7

$$\begin{aligned} \text{a)} \quad & \chi_{g \circ h}(\lambda) = \chi_g(\lambda) \chi_h(\lambda). \\ \text{b)} \quad & \Xi_{g \circ h} = \Xi_g + \Xi_h. \end{aligned}$$

The statement b) is obvious, a) is well-known (see a proof below, Theorem 4.2).

Corollary 2.8 *The \circ -multiplication is well defined as an operation on categorical quotients,*

$$[\operatorname{Coll}_{N_1}(\alpha)] \times [\operatorname{Coll}_{N_2}(\alpha)] \rightarrow [\operatorname{Coll}_{N_1+N_2}(\alpha)]$$

PROOF. Indeed, invariants of $g \circ h$ are determined by invariants of g and h . \square

2.7. The space $\operatorname{Coll}_{\infty}(\alpha)$. Consider the natural map

$$I_N := \operatorname{Mat}(\alpha + N) \rightarrow \operatorname{Mat}(\alpha + N + 1)$$

defined by

$$I_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \chi_{I_N g}(\lambda) &= \chi_g(\lambda); \\ \Xi_{I_N g} &= \Xi_g + \{1\}, \end{aligned}$$

where $\{1\}$ is the point $1 \in K$.

Lemma 2.9 *The induced map $[\text{Coll}_N(\alpha)] \rightarrow [\text{Coll}_{N+1}(\alpha)]$ is an embedding.*

PROOF. The restriction of invariants (2.3)–(2.5) defined on $\text{Mat}(\alpha + N + 1)$ to the subspace $\text{Mat}(\alpha + N)$ gives the same expressions for $\text{Mat}(\alpha + N)$. \square

Thus, we can define a space $[\text{Coll}(\alpha)] = [\text{Coll}_\infty(\alpha)]$ as an inductive limit

$$[\text{Coll}_\infty(\alpha)] = \lim_{N \rightarrow \infty} [\text{Coll}_N(\alpha)].$$

It is equipped with the associative \circ -multiplication.

Characteristic function of an element $g \in [\text{Coll}_\infty(\alpha)]$ can be defined in two equivalent ways. The first way, we write the expression (2.1) for infinite matrix g . The second way. We choose large N such that g has the following representation

$$g = \left(\begin{array}{ccc} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{array} \right) \begin{array}{l} \} \alpha \\ \} N \\ \} \infty \end{array} \quad (2.6)$$

and write the characteristic function for the upper left block of the size $\alpha + N$.

Next, we define the *exceptional divisor* Ξ_g in $\mathbb{P}K^1$. We represent g in the form (2.6), write the exceptional divisor for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and add the point $\lambda = 1$ with multiplicity ∞ (in particular, the multiplicity of 1 always is infinity). Thus, we can regard the 'divisor' as a function

$$\xi : \mathbb{P}K^1 \rightarrow \mathbb{Z}_+ \cup \infty$$

satisfying the following condition.

- a) $\xi(\lambda) = 0$ for all but finite number of λ .
- b) $\xi(1) = \infty$, at all other points ξ is finite.

We reformulate statements obtained above in the following form. Denote by Γ_α the semigroup of rational maps $\chi : \mathbb{P}K^1 \rightarrow \text{Mat}(\alpha)$ regular at the point $\lambda = 0$. Denote by Δ the set of all divisors in the sense described above. We equip Δ with the operation of addition.

Next, consider the subsemigroup $R_\alpha \subset \Gamma \times \Delta$ consisting of pairs (χ, Ξ) such that divisor of the denominator of $\det \chi(\lambda)$ is contained in the divisor Ξ .

Theorem 2.10 *The map $g \mapsto (\chi_g, \Xi_g)$ is an isomorphism of semigroups $[\text{Coll}_\infty(\alpha)]$ and R_α .*

Notice, that the semigroup $[\text{Coll}_\infty(\alpha)]$ itself is not a product of semigroup of characteristic functions and an Abelian semigroup. A similar object appeared in [16], IX.2.

3 The case $\alpha = 1$

3.1. Commutativity.

Theorem 3.1 *The semigroup $[\text{Coll}_\infty(1)]$ is commutative.*

PROOF. Indeed, Γ_1 is commutative, therefore $\Gamma_1 \times \Delta$ is commutative. \square

REMARK. The semigroup $\text{Coll}_\infty(1)$ is not commutative,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and blocks 'd' in the right hand side have different Jordan forms. \square

3.2. Commutativity. Straightforward proof. The proof given below is not necessary in the contexts of this paper. However, it shows that the commutativity in certain sense is a non-obvious fact (in particular, this proof can be modified for proofs of non-commutativity of \circ -products in some cases discussed in [19]).

First, an element of $\text{Coll}_N(1)$ in a general position can be reduced by a conjugation to the form

$$\begin{pmatrix} a & b_1 & b_2 & b_3 & \dots \\ c_1 & \lambda_1 & 0 & 0 & \dots \\ c_2 & 0 & \lambda_2 & 0 & \dots \\ c_3 & 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where λ_j are pairwise distinct. To be short set $N = 2$. Consider two matrices

$$g = \begin{pmatrix} p & b_1 & b_2 \\ c_1 & \lambda_1 & 0 \\ c_2 & 0 & \lambda_2 \end{pmatrix} \quad h = \begin{pmatrix} p & q_1 & q_2 \\ r_1 & \mu_1 & 0 \\ r_2 & 0 & \mu_2 \end{pmatrix}$$

with $\lambda_1, \lambda_2, \mu_1, \mu_2$ being pairwise distinct. We evaluate

$$S = g \circ h = \begin{pmatrix} ap & b_1 & b_2 & aq_1 & aq_2 \\ c_1p & \lambda_1 & 0 & c_1q_1 & c_1q_2 \\ c_2p & 0 & \lambda_2 & c_2q_1 & c_2q_2 \\ r_1 & 0 & 0 & \mu_1 & 0 \\ r_2 & 0 & 0 & 0 & \mu_2 \end{pmatrix}$$

and

$$T = h \circ g = \begin{pmatrix} p & 0 & 0 & q_1 & q_2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ r_1 & 0 & 0 & \mu_1 & 0 \\ r_2 & 0 & 0 & 0 & \mu_2 \end{pmatrix} \begin{pmatrix} a & b_1 & b_2 & 0 & 0 \\ c_1 & \lambda_1 & 0 & 0 & 0 \\ c_2 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} ap & b_1p & b_2p & q_1 & q_2 \\ c_1 & \lambda_1 & 0 & 0 & 0 \\ c_2 & 0 & \lambda_2 & 0 & 0 \\ ar_1 & b_1r_1 & b_2r_1 & \mu_1 & 0 \\ ar_2 & b_1r_2 & b_2r_2 & 0 & \mu_2 \end{pmatrix}$$

Proposition 3.2 *In this notation,*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^{-1} S \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

where

$$U = U_+^{-1} U_d U_-,$$

matrices U_+ , U_- are upper (lower) triangular respectively,

$$U_+ = \begin{pmatrix} 1 & 0 & \frac{c_1 q_1}{\lambda_1 - \mu_1} & \frac{c_1 q_2}{\lambda_1 - \mu_2} \\ 0 & 1 & \frac{c_2 q_1}{\lambda_2 - \mu_1} & \frac{c_2 q_2}{\lambda_2 - \mu_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{b_1 r_1}{\lambda_1 - \mu_1} & \frac{b_2 r_1}{\lambda_2 - \mu_1} & 1 & 0 \\ \frac{b_1 r_2}{\lambda_1 - \mu_2} & \frac{b_2 r_2}{\lambda_2 - \mu_2} & 0 & 1 \end{pmatrix},$$

and U_d is a diagonal matrix with entries

$$p + \frac{q_1 r_1}{\lambda_1 - \mu_1} + \frac{q_2 r_2}{\lambda_1 - \mu_2}, \quad p + \frac{q_1 r_1}{\lambda_2 - \mu_1} + \frac{q_2 r_2}{\lambda_2 - \mu_2},$$

$$\left(a + \frac{b_1 c_1}{\mu_1 - \lambda_1} + \frac{b_2 c_2}{\mu_1 - \lambda_2} \right)^{-1}, \quad \left(a + \frac{b_1 c_1}{\mu_2 - \lambda_1} + \frac{b_2 c_2}{\mu_2 - \lambda_2} \right)^{-1}$$

PROOF. We represent T and S as block matrices,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

of size $(1 + 4) \times (1 + 4)$. We must verify equalities

$$UT_{22} = S_{22}U, \tag{3.1}$$

$$UT_{21} = S_{21}, \quad T_{12} = S_{12}U. \tag{3.2}$$

Represent the first equality in the form

$$U_d(U_- T_{22} U_-^{-1}) = (U_+ S_{22} U_+^{-1}) U_d. \tag{3.3}$$

The matrices U_{\pm} are chosen in such a way that

$$U_- T_{22} U_-^{-1} = U_+ S_{22} U_+^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}$$

Therefore (3.3) holds for any diagonal matrix U_d . It remains to choose U_d to satisfy (3.2). \square

3.3. Linear-fractional transformations.

Proposition 3.3 *Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a nondegenerate 2×2 matrix. Let $g \in \text{Coll}_N(\alpha)$. Let $\gamma g + \delta$ be nondegenerate⁹. Let $\chi_g(s)$ be the characteristic function written in the coordinate $s = \lambda^{-1}$. Then the characteristic function of the colligation*

$$h = (\alpha g + \beta)(\gamma g + \delta)^{-1}$$

is

$$\left(\alpha \chi_g \left(\frac{\alpha s + \beta}{\gamma s + \delta} \right) + \beta \right) \left(\gamma \chi_g \left(\frac{\alpha s + \beta}{\gamma s + \delta} \right) + \delta \right)^{-1}.$$

PROOF. We represent the equation

$$\begin{pmatrix} q \\ sx \end{pmatrix} = h \begin{pmatrix} p \\ x \end{pmatrix}$$

as

$$\begin{pmatrix} \alpha q + \beta p \\ \alpha sx + \beta x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma q + \delta p \\ \gamma sx + \delta x \end{pmatrix}.$$

Passing to the variable $y = (\gamma s + \delta)x$ we get

$$\begin{pmatrix} \alpha q + \beta p \\ (\alpha s + \beta)(\gamma s + \delta)^{-1}x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma q + \delta p \\ x \end{pmatrix}$$

This implies the desired statement. \square

3.4. Non-algebraically closed fields. Now let K be a non-algebraically closed infinite field.

Proposition 3.4 *Let*

$$w(\lambda) = \frac{u(\lambda)}{v(\lambda)}$$

be a rational function on $\mathbb{P}K^1$ such that $v(0) \neq 0$. Then it is a characteristic function of a certain element of $\text{Coll}_{\infty}(1)$.

⁹this is independent on the choice of a representative.

PROOF by induction. Pass to the variable $s = \lambda^{-1}$. We say that degree of $w(s)$ is $\deg v(s)$ (since $s = \infty$ is not a pole of $w(s)$, we have $\deg u(s) \leq \deg v(s)$). For functions of degree 1 the statement is correct. Assume that the statement is correct for functions of degree $< n$. Consider a function $w(s)$ of degree n . Take a linear fractional transformation

$$\tilde{w}(s) := \frac{\alpha w(s) + \beta}{\gamma w(s) + \delta}$$

such that $w(s)$ has a pole at some finite point σ and a zero at some point τ . Then we can decompose $\tilde{w}(s)$:

$$\tilde{w}(s) = \frac{s - \tau}{s - \sigma} y(s),$$

where $y(s)$ is a rational function of degree $< n$. Both factors are characteristic functions, therefore $\tilde{w}(s)$ also is a characteristic function. \square

4 Maps of Bruhat-Tits trees

Now \mathbb{K} is the p -adic field \mathbb{Q}_p and $\mathbb{O} \subset \mathbb{K}$ is the ring of integers. All considerations below can be automatically extended to arbitrary locally compact non-Archimedean fields (few words must be changed).

4.1. Colligations. Denote by $\mathfrak{Coll}_N(\alpha)$ the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over \mathbb{K} of size $\alpha + N$ defined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}^{-1}, \quad \text{where } u \in \text{GL}(N, \mathbb{O}). \quad (4.1)$$

We define \circ -product

$$\mathfrak{Coll}_{N_1}(\alpha) \times \mathfrak{Coll}_{N_2}(\alpha) \rightarrow \mathfrak{Coll}_{N_1+N_2}(\alpha)$$

by the same formula (1.2).

As above we define $\mathfrak{Coll}_\infty(\alpha)$ and the associative \circ -product on $\mathfrak{Coll}_\infty(\alpha)$.

4.2. Bruhat-Tits buildings. Consider a linear space \mathbb{K}^n over \mathbb{K} . A *lattice* R in \mathbb{K}^n is a compact \mathbb{O} -submodule in \mathbb{K}^n such that $\mathbb{K} \cdot R = \mathbb{K}^n$. In other words (see, e.g. [32], [20]), in a certain basis $e_j \in \mathbb{K}^n$, a submodule R has the form $\oplus_j \mathbb{O} e_j$. The space Lat_n of all lattices is a homogeneous space,

$$\text{Lat}_n \simeq \text{GL}(n, \mathbb{K}) / \text{GL}(n, \mathbb{O}).$$

We intend to construct two simplicial complexes BT_n and BT_n^* .

1) Consider an oriented graph, whose vertices are lattices in \mathbb{K}^n . We draw arrow from a vertex R to a vertex T if $T \supset R \supset pT$. If k vertices pairwise are connected by arrows, then we draw a simplex with such vertices. In this way

we get a simplicial complex BT_n , all maximal simplices have dimension n . The group $\text{GL}(n, \mathbb{K})$ acts transitively on the set of all maximal simplices (and also on the set of simplices of each given dimension $j = 0, 1, \dots, n$).

2) Consider a non-oriented graph whose vertices are lattices defined by a dilatation, $R \sim R'$ if $R = \lambda R'$ for some $\lambda \in \mathbb{K}^\times$. Denote

$$\text{Lat}_n^* := \text{Lat}_n / \mathbb{K}^\times.$$

We connect two vertices $R \not\sim T$ by an edge if for some λ we have $pT \subset \lambda R \subset T$. If k vertices pairwise are connected by edges, then we draw a simplex with such vertices. We get a simplicial complex BT_n^* , dimensions of all maximal simplices are $n - 1$. The projective linear group

$$\text{PGL}(n, \mathbb{K}) = \text{GL}(n, \mathbb{K}) / \mathbb{K}^\times$$

acts transitively on the set of all simplices of a given dimension $j = 0, 1, \dots, n - 1$.

We have a natural map

$$\text{BT}_n(\mathbb{K}) \rightarrow \text{BT}_n^*(\mathbb{K}),$$

we send a lattice (a vertex) to the corresponding equivalence class, this induces a map of graphs. Moreover, vertices of a k -dimensional simplex fall to vertices of a simplex of dimension $\leq k$.

These complexes are called '*Bruhat-Tits buildings*', see, e.g., [7], [20]. For $n = 2$ the building $\text{BT}_2(\mathbb{K})$ is an infinite tree, each vertex is an end of $(p + 1)$ edges.

4.3. Construction of characteristic functions. Consider the space $\mathbb{K}^2 = \mathbb{K}^1 \oplus \mathbb{K}^1$. For any lattice $R \subset \mathbb{K}^2$ consider the lattice

$$R \otimes \mathbb{O}^N \subset \mathbb{K}^2 \otimes \mathbb{K}^N = \mathbb{K}^N \oplus \mathbb{K}^N.$$

For a colligation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write the equation

$$\begin{pmatrix} q \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}. \quad (4.2)$$

Consider the set $\chi_g(R)$ of all $q \oplus p \in \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ such that there are $y \oplus x \in R \otimes \mathbb{K}^N$ satisfying the equation (4.2).

Proposition 4.1 a) *The sets $\chi_g(R)$ are lattices.*

b) *If $R, T \in \text{Lat}_2$ are connected by an arrow, then $\chi_g(R)$ and $\chi_g(T)$ are connected by an arrow or coincide.*

c) *A lattice $\chi_g(R)$ depends on the conjugacy class containing g and not on g itself.*

d) $\chi_g(\lambda R) = \lambda \chi_g(R)$ for $\lambda \in \mathbb{K}^\times$.

Proof is given in the next subsection.

Thus χ_g is a map

$$\chi_g : \text{BT}_2(\mathbb{K}) \rightarrow \text{BT}_n(\mathbb{K}).$$

Since it commutes with multiplications by scalars, we get also a well-defined map

$$\chi_g^* : \text{BT}_2(\mathbb{K}) \rightarrow \text{BT}_n^*(\mathbb{K}).$$

4.4. Reformulation of the definition. Consider the space

$$H = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus \mathbb{K}^N \oplus \mathbb{K}^N$$

consisting of vectors with coordinates q, p, y, x . Consider the following subspaces and submodules in H :

- $G \subset H$ is the graph of g ;
- $U = 0 \oplus 0 \oplus \mathbb{K}^N \oplus \mathbb{K}^N$;
- $V = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus 0 \oplus 0$;
- the \mathbb{O} -submodule $S = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus (R \otimes \mathbb{K}^N)$.

Consider the intersection $G \cap S$ and its projection to V along U . The result is $\chi_g(R)$.

PROOF OF PROPOSITION 4.1. The statements a), b), d) follow from new version of the definition, c) follows from $\text{GL}(N, \mathbb{O})$ -invariance of $R \otimes \mathbb{O}^N$. \square

4.5. Products. Now we wish to obtain an analog of Theorem 2.7. For this purpose, we need a definition of multiplication of lattices.

Let $S, T \subset \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ be lattices. We define their *product* ST as the set of all $u \oplus w \in \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ such that there is $v \in \mathbb{K}^\alpha$ satisfying $u \oplus v \in S, v \oplus w \in T$. This is the usual product of relations (or multi-valued maps), see, e.g., [16].

Theorem 4.2

$$\chi_{g \circ h}(S) = \chi_g(S) \chi_h(S).$$

PROOF. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let $r = \chi_g(S)q, q = \chi_h(S)p$. Then there are z, y, y', x such that

$$y \oplus x \in R \otimes \mathbb{O}^{N_1}, \quad y' \oplus x' \in R \otimes \mathbb{O}^{N_2}$$

satisfying

$$\begin{pmatrix} r \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ x \end{pmatrix}, \quad \begin{pmatrix} q \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p \\ x' \end{pmatrix}.$$

Then

$$\begin{pmatrix} r \\ y \\ y' \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ x \\ y' \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ \gamma & 0 & \delta \end{pmatrix} \begin{pmatrix} q \\ x \\ x' \end{pmatrix}.$$

This proves the desired statement. \square

Consider the natural projection

$$\text{pr} : \mathfrak{Coll}_N(\alpha) \rightarrow \text{Coll}_n(\alpha).$$

Formally, we have two characteristic functions of an element of $\mathfrak{Coll}_N(\alpha)$, one is defined on \mathbb{PK}^1 , another on $\text{BT}_2(\mathbb{K})$. In fact, the second function is the value of the first on the boundary of the building. Now we intend to explain this.

4.6. Convergence of lattices to subspaces. We say that a sequence of lattices $R_j \in \text{Lat}_n$ converges to a subspace $L \subset \mathbb{K}^n$ if

- a) For each ε for sufficiently large j a lattice R_j is contained in the ε -neighborhood of L
- b) For each compact set $S \subset L$ we have $R_j \cap L \subset S$ for sufficiently large j .

Proposition 4.3 *Let a sequence $R_j \in \text{Lat}_n$ converge to a subspace $L \subset \mathbb{O}^n$. Let $M \subset \mathbb{O}^n$ be a subspace. Let $\pi : \mathbb{O}^n \rightarrow \mathbb{O}^n/L$ be the natural projection. Then*

- a) $R_j \cap M$ converges to $L \cap M$.
- b) $\pi(R_j)$ converges to $\pi(M)$.

The statement is obvious.

We say that a sequence $R_j^* \in \text{Lat}^*$ converges to a subspace L if we have a convergence $R_j \rightarrow L$ for some representatives of R_j^* . Notice that a sequence R_j^* can have many limits in this sense^{10,11}. However a limit subspace of a given dimension is unique.

4.7. Boundary values.

Proposition 4.4 *Let $g \in \mathfrak{Coll}_N(\alpha)$. Let $\lambda \in \mathbb{PK}^1$ be a nonsingular point of the characteristic function $\chi_{\text{pr}(g)}(\lambda)$ defined on \mathbb{PK}^1 . Let L be the line in \mathbb{K}^2 corresponding λ . Let $R_j \in \text{Lat}_n(\mathbb{K})$ converges to ℓ . Then $\chi_g(R_j)$ converges to $\chi_{\text{pr}(g)}(\lambda)$*

PROOF. The statement follows from Subsection 4.4 and Proposition 4.3. \square

4.8. Rational maps of Bruhat–Tits trees.

Corollary 4.5 *Any rational map $\mathbb{PK}^1 \rightarrow \mathbb{PK}^1$ can be extended to a continuous map of Bruhat–Tits trees, such that image of a vertex is a vertex and image of an edge is an edge or a vertex.*

PROOF. Represent a rational map as a characteristic function of a colligation $q \in \text{Coll}_\infty(1)$. We take a colligation $g \in \mathfrak{Coll}_\infty(1)$ such that $\text{pr}(g) = q$, and take the corresponding map $\text{BT}_2^*(\mathbb{K}) \rightarrow \text{BT}_2^*(\mathbb{K})$.

¹⁰Moreover, 0 and \mathbb{O}^n are limits of all sequences according our definition.

¹¹See [17].

5 Rational maps of buildings.

5.1. m -colligations. Fix $\alpha \geq 0$, $m \geq 1$. Let $N > 0$. Consider the space $\text{Mat}(\alpha + mN, \mathbb{K})$ of block matrices of size $\alpha + N + \dots + N$. Denote by $\mathbf{Coll}_N(\alpha|m)$ the set of such matrices up to the equivalence

$$\begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix} \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix}^{-1},$$

where $u \in \text{GL}(N, \mathbb{O})$. (5.1)

We define a multiplication

$$\mathbf{Coll}_{N_1}(\alpha|m) \times \mathbf{Coll}_{N_2}(\alpha|m) \rightarrow \mathbf{Coll}_{N_1+N_2}(\alpha|m)$$

by

$$\begin{aligned} & \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \circ \begin{pmatrix} p & q_1 & \dots & q_m \\ r_1 & t_{11} & \dots & t_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ r_m & t_{m1} & \dots & t_{mm} \end{pmatrix} = \\ & = \begin{pmatrix} a & b_1 & 0 & \dots & b_m & 0 \\ c_1 & d_{11} & 0 & \dots & d_{1m} & 0 \\ 0 & 0 & 1_{N_2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_m & d_{m1} & 0 & \dots & d_{mm} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1_{N_2} \end{pmatrix} \begin{pmatrix} p & 0 & q_1 & \dots & 0 & q_m \\ 0 & 1_{N_1} & 0 & \dots & 0 & 0 \\ r_1 & 0 & t_{11} & \dots & 0 & t_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1_{N_1} & 0 \\ r_m & t_{m1} & 0 & \dots & 0 & t_{mm} \end{pmatrix} = \\ & = \begin{pmatrix} a & b_1 & aq_1 & \dots & b_m & aq_m \\ c_1p & d_{11} & c_1q_1 & \dots & d_{1m} & c_1q_m \\ r_1 & 0 & t_{11} & \dots & 0 & t_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_mp & d_{m1} & c_mq_1 & \dots & d_{mm} & c_mq_m \\ r_m & 0 & t_{m1} & \dots & 0 & t_{mm} \end{pmatrix} \end{aligned}$$

5.2. Characteristic functions. For a lattice $R \in \text{Lat}_{2m}(\mathbb{K})$ consider the lattice

$$R \otimes \mathbb{O}^N \subset \mathbb{K}^{2m} \otimes \mathbb{K}^N = (\mathbb{K}^m \otimes \mathbb{K}^N) \oplus (\mathbb{K}^m \otimes \mathbb{K}^N)$$

For $g \in \text{Mat}(\alpha + km)$ we write the following equation

$$\begin{pmatrix} q \\ y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} p \\ x_1 \\ \vdots \\ x_m \end{pmatrix}, \quad (5.2)$$

where p, q range in \mathbb{K}^α , and $x_j, y_j \in \mathbb{K}^N$. Denote by $\chi_g(R)$ the set of all $q \oplus p \in \mathbb{K}^{2\alpha}$ such that there exists $y \oplus x \in R \otimes \mathbb{O}^N$, for which equality (5.2) holds.

Theorem 5.1 a) $\chi_g(R)$ is a lattice in $\mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$.

b) The characteristic function $\chi_g(R)$ is an invariant of the equivalence (5.1).

c) The map $\chi_g : \text{Lat}_{2m} \rightarrow \text{Lat}_{2\alpha}$ induces maps

$$\text{BT}_{2m} \rightarrow \text{BT}_{2\alpha}, \quad \text{BT}_{2m}^* \rightarrow \text{BT}_{2\alpha}^*.$$

c) For any $g \in \mathfrak{Coll}_{N_1}(\alpha|m)$, $h \in \mathfrak{Coll}_{N_2}(\alpha|m)$, the following identity holds

$$\chi_{g \circ h}(R) = \chi_g(R) \chi_h(R).$$

PROOFS repeats the proofs given above for $m = 1$. See, also a more sophisticated objects in [23]. \square

5.3. Extension to the boundary. Next (see [18], [21]), we extend characteristic functions to the distinguished boundaries of buildings. Let $S \in \text{Mat}(m, \mathbb{K})$. Again write equation (5.2). We say $q = \chi_g(S)p$ if there exists y such that $q, p, y, x = Sy$ satisfy the equation (5.2). In other words,

$$\chi_g(S) = a + b\tilde{S}(1 - d\tilde{S})^{-1}c,$$

where $\tilde{S} = S \otimes 1_N$,

$$\tilde{S} = \begin{pmatrix} s_{11} \cdot 1_N & \dots & s_{1m} \cdot 1_N \\ \vdots & \ddots & \vdots \\ s_{m1} \cdot 1_N & \dots & s_{mm} \cdot 1_N \end{pmatrix}$$

Theorem 5.2 a) For any $g \in \mathfrak{Coll}_{N_1}(\alpha|m)$, $h \in \mathfrak{Coll}_{N_2}(\alpha|m)$,

$$\chi_{g \circ h}(S) = \chi_g(S) \chi_h(S).$$

b) If a sequence of lattices $R_j \in \text{Lat}(\mathbb{K}^m \oplus \mathbb{K}^m)$ converges to the graph of S , then $\chi_g(R_j)$ converges to $\chi_g(S)$.

Proof is the same as above for $m = 1$.

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